Chapter 6

Some Types of HyperNeutrosophic Set (4): Cubic, Trapozoidal, q-Rung Orthopair, Overset, Underset, and Offset

Takaaki Fujita [1] * and Florentin Smarandache[2],
[1]* Independent Researcher, Shinjuku, Shinjuku-ku, Tokyo, Japan. t171d603@gunma-u.ac.jp
[2] University of New Mexico, Gallup Campus, NM 87301, USA. smarand@unm.edu

Abstract

This paper builds upon the foundational work presented in [38–40]. The Neutrosophic Set provides a comprehensive mathematical framework for managing uncertainty, defined by three membership functions: truth, indeterminacy, and falsity. Recent advancements have introduced extensions such as the Hyperneutrosophic Set and the SuperHyperneutrosophic Set, which are specifically designed to address increasingly complex and multidimensional problems. The formal definitions of these sets are available in [30].

In this paper, we extend the Neutrosophic Cubic Set, Trapezoidal Neutrosophic Set, q-Rung Orthopair Neutrosophic Set, Neutrosophic Overset, Neutrosophic Underset, and Neutrosophic Offset using the frameworks of the Hyperneutrosophic Set and the SuperHyperneutrosophic Set. Furthermore, we briefly examine their properties and potential applications.

Keywords: Set Theory, SuperhyperNeutrosophic set, Neutrosophic Set, HyperNeutrosophic set

1 Preliminaries and Definitions

This section provides an overview of the fundamental concepts and definitions essential for the discussions in this paper. The analysis utilizes classical set-theoretic operations and extends them into advanced frameworks. For readers seeking a deeper understanding of foundational set theory, resources such as [16, 52, 55, 60] are recommended. Additionally, the referenced literature offers a comprehensive exploration of the principles and applications of Neutrosophic Sets.

1.1 Neutrosophic, HyperNeutrosophic, and n-SuperHyperNeutrosophic Sets

To address uncertainty, vagueness, and imprecision in decision-making processes, numerous set-theoretic frameworks have been developed. These frameworks include Fuzzy Sets, which were introduced in foundational works such as those by Zadeh [105–109]. Another prominent framework is Intuitionistic Fuzzy Sets, extensively studied by Atanassov and others [5–10]. Vague Sets, introduced and developed by researchers, also contribute significantly to this domain [1, 11, 49, 63, 74].

More recently, Plithogenic Sets, as proposed and expanded by Smarandache, have gained attention for their ability to model complex scenarios involving contradictions and multi-dimensional uncertainty [18, 24, 26–28, 36, 37, 46, 85, 87, 88]. Soft Sets, as introduced by Molodtsov and further studied by other scholars, provide a flexible mathematical tool for handling uncertainty [50, 64, 67].

Additionally, Hypersoft Sets, an extension of Soft Sets, have been explored in various applications by Smarandache [20, 31, 45, 86]. Neutrosophic Sets, first introduced by Smarandache, offer a powerful means of capturing indeterminacy, allowing for more nuanced decision-making models [21, 22, 25, 35, 41–44, 47, 48, 79, 80, 94]. Neutrosophic Sets generalize Fuzzy Sets by introducing an additional component: indeterminacy, alongside truth and falsity [77–80]. This enhancement allows for a richer and more precise representation of uncertainty and ambiguity.

To address even more complex scenarios, the HyperNeutrosophic Sets and *n*-SuperHyperNeutrosophic Sets have been developed. These advanced models are particularly suited for high-dimensional and intricate problem spaces [19, 30].

Definition 1.1 (Base Set). A *base set* S is the foundational set from which complex structures such as powersets and hyperstructures are derived. It is formally defined as:

$$S = \{x \mid x \text{ is an element within a specified domain}\}.$$

All elements in constructs like $\mathcal{P}(S)$ or $\mathcal{P}_n(S)$ originate from the elements of S.

Definition 1.2 (Powerset). [26, 73] The *powerset* of a set S, denoted $\mathcal{P}(S)$, is the collection of all possible subsets of S, including both the empty set and S itself. Formally, it is expressed as:

$$\mathcal{P}(S) = \{A \mid A \subseteq S\}.$$

Definition 1.3 (n-th Powerset). (cf. [17, 26, 32, 76, 91])

The n-th powerset of a set H, denoted $P_n(H)$, is defined iteratively, starting with the standard powerset. The recursive construction is given by:

$$P_1(H) = P(H), \quad P_{n+1}(H) = P(P_n(H)), \quad \text{for } n \geq 1.$$

Similarly, the n-th non-empty powerset, denoted $P_n^*(H)$, is defined recursively as:

$$P_1^*(H) = P^*(H), \quad P_{n+1}^*(H) = P^*(P_n^*(H)).$$

Here, $P^*(H)$ represents the powerset of H with the empty set removed.

Definition 1.4 (Neutrosophic Set). [79, 80] Let X be a non-empty set. A *Neutrosophic Set (NS) A* on X is characterized by three membership functions:

$$T_A : X \rightarrow [0,1], \quad I_A : X \rightarrow [0,1], \quad F_A : X \rightarrow [0,1],$$

where for each $x \in X$, the values $T_A(x)$, $I_A(x)$, and $F_A(x)$ represent the degrees of truth, indeterminacy, and falsity, respectively. These values satisfy the following condition:

$$0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3.$$

Definition 1.5 (HyperNeutrosophic Set). (cf. [19, 30, 33, 34, 84]) Let X be a non-empty set. A *HyperNeutrosophic Set (HNS)* $\tilde{A}$ on X is a mapping:

$$\tilde{\mu} : X \rightarrow \mathcal{P}([0,1]^3),$$

where $\mathcal{P}([0,1]^3)$ is the family of all non-empty subsets of the unit cube $[0,1]^3$. For each $x \in X$, $\tilde{\mu}(x) \subseteq [0,1]^3$ is a set of neutrosophic membership triplets (T, I, F) that satisfy:

$$0 \leq T + I + F \leq 3.$$

Definition 1.6 (n-SuperHyperNeutrosophic Set). (cf. [19, 30, 33, 34, 84]) Let X be a non-empty set. An *n-SuperHyperNeutrosophic Set (n-SHNS)* is a recursive generalization of Neutrosophic Sets and HyperNeutrosophic Sets. It is defined as a mapping:

$$\tilde{A}_n : \mathcal{P}_n(X) \rightarrow \mathcal{P}_n([0,1]^3),$$

where:

- $\mathcal{P}_1(X) = \mathcal{P}(X)$, the power set of X, and for $k \geq 2$,

$$\mathcal{P}_k(X) = \mathcal{P}(\mathcal{P}_{k-1}(X)),$$

  representing the k-th nested family of non-empty subsets of X.

- $\mathcal{P}_n([0,1]^3)$ is defined similarly for the unit cube $[0,1]^3$.

For each $A \in \mathcal{P}_n(X)$ and $(T, I, F) \in \tilde{A}_n(A)$, the following condition is satisfied:

$$0 \leq T + I + F \leq 3,$$

where T, I, F represent the degrees of truth, indeterminacy, and falsity for the n-th level subsets of X.

2 Results of This Paper

This section outlines the main results presented in this paper.

2.1 Neutrosophic Cubic Set

A Neutrosophic Cubic Set (NCS) combines Interval Neutrosophic Sets and Neutrosophic Sets, representing uncertainty through interval and point-based truth, indeterminacy, and falsity values [12,14,51,56,70,96,110].

Definition 2.1 (Neutrosophic Cubic Set (NCS)). [3,56] Let X be a non-empty set. A *Neutrosophic Cubic Set (NCS) A* in X is a pair $A = (A_{INS}, A_{NS})$, where:

- $A_{INS} = \{\langle x, T_A(x), I_A(x), F_A(x) \rangle \mid x \in X\}$ is an *Interval Neutrosophic Set (INS)* in X. For each $x \in X$, $T_A(x) = [T_A^-, T_A^+]$, $I_A(x) = [I_A^-, I_A^+]$, $F_A(x) = [F_A^-, F_A^+]$, where $T_A, I_A, F_A \subseteq [0,1]$.

- $A_{NS} = \{\langle x, T_A(x), I_A(x), F_A(x) \rangle \mid x \in X\}$ is a *Neutrosophic Set (NS)* in X. Here, $T_A, I_A, F_A : X \to [0,1]$, satisfying $0 \le T_A(x) + I_A(x) + F_A(x) \le 3$ for all $x \in X$.

The pair $A = (A_{INS}, A_{NS})$ generalizes the notions of Interval Neutrosophic Sets and Neutrosophic Sets, allowing for a hybrid representation of uncertainty.

Remark 2.2 (Neutrosophic Cubic membership domain). For convenience, define the *Neutrosophic Cubic membership domain* $C \subseteq [0,1]^9$ by:

$$
C = \left\{ (T^-, T^+, I^-, I^+, F^-, F^+, T, I, F) \in [0,1]^9 : \begin{array}{l} 0 \le T^- \le T \le T^+ \le 1, \\ 0 \le I^- \le I \le I^+ \le 1, \\ 0 \le F^- \le F \le F^+ \le 1, \\ (T^- + I^- + F^-) \le 3, \quad (T^+ + I^+ + F^+) \le 3, \quad (T + I + F) \le 3 \end{array} \right\}.
$$

Each 9-tuple in C represents both *interval* membership (the triple intervals $[T^-, T^+]$, $[I^-, I^+]$, $[F^-, F^+]$) and *point* membership (T, I, F), subject to usual neutrosophic constraints.

Definition 2.3 (HyperNeutrosophic Cubic Set (HNCS)). Let X be a non-empty set, and let $\mathcal{P}(C)$ be the family of all non-empty subsets of the domain $C \subseteq [0,1]^9$ (as defined above). A *HyperNeutrosophic Cubic Set (HNCS)* $\widetilde{N}$ on X is a mapping

$$\widetilde{N} : X \longrightarrow \mathcal{P}(C),$$

where for each $x \in X$, $\widetilde{N}(x)$ is a *set* of 9-tuples

$$\left(T^-, T^+, I^-, I^+, F^-, F^+, T, I, F\right) \in C$$

satisfying the constraints for Neutrosophic Cubic membership (i.e. $T^- \le T \le T^+$, $T^- + I^- + F^- \le 3$, etc.).

Hence, each point x may have *multiple* possible cubic memberships, capturing a range (hyper-set) of intervals plus point-based membership data.

Theorem 2.4. *Every Neutrosophic Cubic Set is a special case of a HyperNeutrosophic Cubic Set.*

Proof. A *Neutrosophic Cubic Set (NCS) A* over X assigns each $x \in X$ a single pair $(A_{INS}(x), A_{NS}(x))$ of interval membership plus point membership. Concretely, it can be described by a single 9-tuple

$$\left(T_A^-(x), T_A^+(x), I_A^-(x), I_A^+(x), F_A^-(x), F_A^+(x), T_A(x), I_A(x), F_A(x)\right) \in C.$$

In Definition 2.3, an HNCS is a mapping $\widetilde{N} : X \to \mathcal{P}(C)$. We embed A by letting

$$\widetilde{N}(x) = \left\{\left(T_A^-(x), T_A^+(x), I_A^-(x), I_A^+(x), F_A^-(x), F_A^+(x), T_A(x), I_A(x), F_A(x)\right)\right\},$$

a *singleton* in C. Since all constraints on intervals and points match those in the domain C, A is reproduced exactly. Thus, every NCS is a degenerate (single membership) version of an HNCS. □

Theorem 2.5. *Every HyperNeutrosophic Set is a special case of a HyperNeutrosophic Cubic Set by collapsing the interval portion to a single point.*

Proof. A *HyperNeutrosophic Set (HNS)* $\tilde{A}$ maps each $x \in X$ to a subset of $[0, 1]^3$ with $T + I + F \leq 3$. In Definition 2.3, we use $C \subseteq [0, 1]^9$. If we force $T^- = T = T^+$, $I^- = I = I^+$, $F^- = F = F^+$, then the 9-tuple

$$(T^-, T^+, I^-, I^+, F^-, F^+, T, I, F)$$

collapses to $(T, T, T, I, I, I, F, F, F)$ with $T + I + F \leq 3$. This effectively recovers a 3D membership (T,I,F). Formally, define

$$\widetilde{N}(x) = \Big\{ (T, T, T, I, I, I, F, F, F) \ \mid \ (T, I, F) \in \tilde{A}(x) \Big\}.$$

Hence, ignoring the intervals (merging them with the single values) yields a standard hyperneutrosophic membership. Therefore, an HNS is embedded in an HNCS by collapsing intervals to single points. $\qquad\square$

Definition 2.6 (n-SuperHyperNeutrosophic Cubic Set (n-SHNCS))**.** Let X be a non-empty set. Define:

$$\mathcal{P}_1(X) = \mathcal{P}(X), \quad \mathcal{P}_k(X) = \mathcal{P}\big(\mathcal{P}_{k-1}(X)\big) \quad (k \geq 2).$$

Likewise, define $\mathcal{P}_n(C)$ as the n-th nested power set of the cubic domain $C \subseteq [0, 1]^9$ from above. An *n-SuperHyperNeutrosophic Cubic Set (n-SHNCS)* is a mapping

$$\widetilde{N}_n : \mathcal{P}_n(X) \ \longrightarrow \ \mathcal{P}_n(C),$$

such that for each $A \in \mathcal{P}_n(X)$, $\widetilde{N}_n(A) \subseteq C$. Concretely, each n-th level subset A in X is assigned a *set* of 9-tuples

$$\big(T^-, T^+, I^-, I^+, F^-, F^+, T, I, F\big) \ \in \ C,$$

all obeying the neutrosophic cubic constraints (interval plus point membership).

Theorem 2.7. *Every HyperNeutrosophic Cubic Set is a special case of an n-SuperHyperNeutrosophic Cubic Set for $n = 1$.*

Proof. A *HyperNeutrosophic Cubic Set (HNCS)* $\widetilde{N}$ is a mapping $X \to \mathcal{P}(C)$. In Definition 2.6, if we set $n = 1$, we get

$$\widetilde{N}_1 : \mathcal{P}_1(X) = \mathcal{P}(X) \ \longrightarrow \ \mathcal{P}_1(C) = \mathcal{P}(C).$$

We define

$$\widetilde{N}_1(\{x\}) = \widetilde{N}(x), \quad \widetilde{N}_1(A) = \varnothing \quad \text{for } A \neq \{x\}.$$

Hence, for singletons $A = \{x\}$, $\widetilde{N}_1(\{x\})$ recovers exactly the membership set $\widetilde{N}(x)$ in C. The same constraints remain. Therefore, every HNCS is included in an n-SuperHyperNeutrosophic Cubic Set with $n = 1$. $\qquad\square$

Theorem 2.8. *Every n-SuperHyperNeutrosophic Set is a special case of an n-SuperHyperNeutrosophic Cubic Set by collapsing the interval membership to single points.*

Proof. An *n-SuperHyperNeutrosophic Set (SHNS)* $\tilde{A}_n$ maps each $A \in \mathcal{P}_n(X)$ to subsets of $[0, 1]^3$, each triple (T, I, F) satisfying $T + I + F \leq 3$. In Definition 2.6, $\widetilde{N}_n(A)$ is a subset of the domain $C \subseteq [0, 1]^9$. To recover an SHNS from n-SHNCS, we identify $(T^-, T^+, I^-, I^+, F^-, F^+, T, I, F)$ with $(T, T, I, I, F, F, T, I, F)$ in which $T^- = T^+ = T$, $I^- = I^+ = I$, and $F^- = F^+ = F$. Then $T + I + F \leq 3$ is the standard constraint. Formally:

$$\widetilde{N}_n(A) = \Big\{ (T, T, I, I, F, F, T, I, F) \ \mid \ (T, I, F) \in \tilde{A}_n(A) \Big\}.$$

Hence, ignoring intervals or collapsing them to single points recovers a 3D membership in $[0, 1]^3$. Thus, an SHNS is embedded in n-SHNCS by dropping the interval portion. $\qquad\square$

2.2 Trapezoidal Neutrosophic Set

A Trapezoidal Neutrosophic Set (TNS) utilizes trapezoidal fuzzy numbers to represent truth, indeterminacy, and falsity memberships, enabling advanced modeling of uncertainty [4, 13, 57, 59, 101, 102]. A closely related concept is the Trapezoidal Fuzzy Set [61, 66, 100, 103, 104].

Definition 2.9 (Trapezoidal Neutrosophic Set). [101] A *Trapezoidal Neutrosophic Set (TNS) A* in a universe of discourse X is defined as:
$$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle \mid x \in X\},$$
where:
$$T_A(x) = (t_1, t_2, t_3, t_4), \quad I_A(x) = (i_1, i_2, i_3, i_4), \quad F_A(x) = (f_1, f_2, f_3, f_4),$$
are *trapezoidal fuzzy numbers* that represent the truth-membership, indeterminacy-membership, and falsity-membership functions, respectively. These functions satisfy the following conditions:
$$t_1 \le t_2 \le t_3 \le t_4, \quad i_1 \le i_2 \le i_3 \le i_4, \quad f_1 \le f_2 \le f_3 \le f_4,$$
and
$$0 \le T_A(x) + I_A(x) + F_A(x) \le 3 \quad \forall x \in X.$$

Each trapezoidal membership function is defined piecewise:

$$T_A(x) = \begin{cases} \frac{x-t_1}{t_2-t_1}, & t_1 \le x \le t_2, \\ 1, & t_2 \le x \le t_3, \\ \frac{t_4-x}{t_4-t_3}, & t_3 \le x \le t_4, \\ 0, & \text{otherwise.} \end{cases}$$

The indeterminacy-membership $I_A(x)$ and falsity-membership $F_A(x)$ follow similar definitions with their respective parameters.

Remark 2.10 (Trapezoidal Neutrosophic domain). To handle trapezoids and the neutrosophic constraint, define the *Trapezoidal Neutrosophic domain*:
$$\mathcal{T} \subseteq ([0,1]^4)^3$$
where each triple $\big((t_1, t_2, t_3, t_4), (i_1, i_2, i_3, i_4), (f_1, f_2, f_3, f_4)\big)$ must satisfy
$$t_1 \le t_2 \le t_3 \le t_4, \quad i_1 \le i_2 \le i_3 \le i_4, \quad f_1 \le f_2 \le f_3 \le f_4,$$
and possibly a constraint like $T_A(x) + I_A(x) + F_A(x) \le 3$ in an integrated sense (though exact interpretation can vary). For simplicity, we can embed the trapezoid-based membership directly, assuming each trapezoid is in $[0,1]^4$ with ascending coordinates.

Definition 2.11 (Trapezoidal HyperNeutrosophic Set (THNS)). Let X be a non-empty set, and let $\mathcal{P}(\mathcal{T})$ be the family of all non-empty subsets of the trapezoidal domain $\mathcal{T} \subseteq ([0,1]^4)^3$. A *Trapezoidal HyperNeutrosophic Set (THNS) $\widetilde{T}$* on X is a mapping
$$\widetilde{T} : X \longrightarrow \mathcal{P}(\mathcal{T}),$$
such that for each $x \in X$, $\widetilde{T}(x)$ is a *set* of trapezoidal triplets
$$\big((t_1, t_2, t_3, t_4), (i_1, i_2, i_3, i_4), (f_1, f_2, f_3, f_4)\big) \in \mathcal{T},$$
capturing multiple possible trapezoidal membership functions for truth, indeterminacy, and falsity. Each triple of trapezoids is typically constrained by $0 \le t_1 \le t_2 \le t_3 \le t_4 \le 1$, etc., and respects a neutrosophic boundary (e.g. up to ≤ 3 in some integrated sense).

Theorem 2.12. *Every Trapezoidal Neutrosophic Set is a special case of a Trapezoidal HyperNeutrosophic Set.*

Proof. A *Trapezoidal Neutrosophic Set (TNS) A* assigns each $x \in X$ exactly one triple of trapezoids $(T_A(x), I_A(x), F_A(x)) \in ([0,1]^4)^3$. In Definition 2.11, we define $\widetilde{T}(x) \subseteq \mathcal{T}$. We embed A by letting
$$\widetilde{T}(x) = \Big\{\big(T_A(x), I_A(x), F_A(x)\big)\Big\},$$
a singleton set. This precisely recovers the TNS membership. Hence, every TNS is embedded in THNS as a degenerate (single membership) case. □

Theorem 2.13. *Every HyperNeutrosophic Set is a special case of a Trapezoidal HyperNeutrosophic Set by collapsing trapezoids to single numeric values.*

Proof. A *HyperNeutrosophic Set (HNS)* $\tilde{A}$ maps $x \in X$ to subsets of $[0,1]^3$, each triple (T, I, F) with $T + I + F \le 3$. In Definition 2.11, each membership is in $\mathcal{T} \subseteq ([0,1]^4)^3$. If we set $t_1 = t_2 = t_3 = t_4 = T$, $i_1 = i_2 = i_3 = i_4 = I$, $f_1 = f_2 = f_3 = f_4 = F$, each trapezoid degenerates to a single point. Formally:

$$\widetilde{T}(x) = \Big\{ \big((T,T,T,T), (I,I,I,I), (F,F,F,F)\big) \;\Big|\; (T,I,F) \in \tilde{A}(x) \Big\}.$$

Thus, ignoring the trapezoidal range merges the set into numeric values. Hence, an HNS emerges as a special (collapsed trapezoid) case of THNS. □

Definition 2.14 (Trapezoidal n-SuperHyperNeutrosophic Set (T-n-SHNS)). Let X be a non-empty set. Define:

$$\mathcal{P}_1(X) = \mathcal{P}(X), \quad \mathcal{P}_k(X) = \mathcal{P}\big(\mathcal{P}_{k-1}(X)\big) \quad (k \ge 2).$$

Similarly, let $\mathcal{P}_n(\mathcal{T})$ denote the n-th nested power set of the trapezoidal domain $\mathcal{T} \subseteq ([0,1]^4)^3$. A *Trapezoidal n-SuperHyperNeutrosophic Set (T-n-SHNS)* is a mapping

$$\widetilde{T}_n : \mathcal{P}_n(X) \longrightarrow \mathcal{P}_n(\mathcal{T}),$$

meaning for each $A \in \mathcal{P}_n(X), \widetilde{T}_n(A) \subseteq \mathcal{T}$. Concretely, each n-th level subset A is assigned a *set* of trapezoidal membership triples

$$\big(T_A(x), I_A(x), F_A(x)\big) \in \big([0,1]^4\big)^3,$$

satisfying the trapezoidal ordering constraints and a neutrosophic boundary (e.g. up to ≤ 3 in some integrated sense).

Theorem 2.15. *Every Trapezoidal HyperNeutrosophic Set is a special case of a Trapezoidal n-SuperHyperNeutrosophic Set (T-n-SHNS) for $n = 1$.*

Proof. A *Trapezoidal HyperNeutrosophic Set (THNS)* $\widetilde{T}$ (Definition 2.11) is a mapping $X \rightarrow \mathcal{P}(\mathcal{T})$. In Definition 2.14, for $n = 1$ we have

$$\widetilde{T}_1 : \mathcal{P}_1(X) = \mathcal{P}(X) \rightarrow \mathcal{P}_1(\mathcal{T}) = \mathcal{P}(\mathcal{T}).$$

We embed $\widetilde{T}$ by defining:

$$\widetilde{T}_1(\{x\}) := \widetilde{T}(x), \quad \widetilde{T}_1(A) = \varnothing \quad (\text{for } A \ne \{x\}).$$

Hence, each singleton $\{x\} \subseteq X$ recovers exactly $\widetilde{T}(x)$. Thus, $\widetilde{T}_1$ is a T-1-SHNS that coincides with the THNS $\widetilde{T}$. □

Theorem 2.16. *Every n-SuperHyperNeutrosophic Set is a special case of a Trapezoidal n-SuperHyperNeutrosophic Set by collapsing trapezoids to single points.*

Proof. An *n-SuperHyperNeutrosophic Set (SHNS)* $\tilde{A}_n$ maps $\mathcal{P}_n(X)$ to subsets of $[0,1]^3$. In Definition 2.14, T-n-SHNS uses $\mathcal{T} \subseteq (([0,1]^4)^3)$. If we make each trapezoid degenerate, e.g. $t_1 = t_2 = t_3 = t_4 = T$, etc., we effectively recover single numeric values (T, I, F). Formally:

$$\widetilde{T}_n(A) = \Big\{ \big((T,T,T,T), (I,I,I,I), (F,F,F,F)\big) \;\Big|\; (T,I,F) \in \tilde{A}_n(A) \Big\}.$$

Hence, ignoring the trapezoidal range merges the membership into single numeric triplets. Thus, any n-SuperHyperNeutrosophic Set is included in T-n-SHNS by collapsing the trapezoids to single points. □

2.3 q-Rung Orthopair Neutrosophic Set

A q-Rung Orthopair Neutrosophic Set (q-RONS) generalizes orthopair sets, constraining q-th powers of truth, indeterminacy, and falsity to sum ≤ 2 [75, 97, 98]. Related concepts include the q-Rung Orthopair Fuzzy Set, among others [2, 15, 29, 53, 54, 58, 62, 69, 71, 72, 93, 99].

Definition 2.17 (q-Rung Orthopair Neutrosophic Set). [75, 98] Let U be a universal set. A *q-Rung Orthopair Neutrosophic Set (q-RONS)* is defined as:

$$A = \{(x, T_A(x), I_A(x), F_A(x)) \mid x \in U\},$$

where $T_A(x)$, $I_A(x)$, and $F_A(x)$ are the truth-membership, indeterminacy-membership, and falsity-membership degrees, respectively. These satisfy:

1. $T_A(x), I_A(x), F_A(x) \in [0, 1]$,

2.
$$[T_A(x)]^q + [I_A(x)]^q + [F_A(x)]^q \leq 2, \quad q > 0.$$

Definition 2.18 (q-Rung Orthopair HyperNeutrosophic Set (q-RHNS)). Let U be a non-empty set, and let $q > 0$. A *q-Rung Orthopair HyperNeutrosophic Set (q-RHNS)* on U is a mapping

$$\widetilde{Q} : U \longrightarrow \mathcal{P}([0, 1]^3),$$

where for each $x \in U$, $\widetilde{Q}(x) \subseteq [0, 1]^3$ is a *set* of triplets (T, I, F), each triplet satisfying

$$T^q + I^q + F^q \leq 2, \quad (T, I, F) \in [0, 1]^3.$$

Theorem 2.19. *Every q-Rung Orthopair Neutrosophic Set is a special case of a q-Rung Orthopair HyperNeutrosophic Set.*

Proof. A *q-Rung Orthopair Neutrosophic Set (q-RONS)* A on U assigns each $x \in U$ exactly one triplet $(T_A(x), I_A(x), F_A(x)) \in [0, 1]^3$ with $(T_A(x))^q + (I_A(x))^q + (F_A(x))^q \leq 2$. In Definition 2.18, we let each x map to a *set* of triplets. So define:

$$\widetilde{Q}(x) = \left\{ \left(T_A(x), I_A(x), F_A(x) \right) \right\},$$

a singleton set. The same q-rung condition persists. Hence, each q-RONS is naturally embedded in the q-RHNS framework as a degenerate (singleton) membership set. □

Theorem 2.20. *Every HyperNeutrosophic Set can be viewed as a special case of a q-Rung Orthopair HyperNeutrosophic Set by setting $q = 1$ or adjusting membership sums.*

Proof. A *HyperNeutrosophic Set (HNS)* $\tilde{A}$ maps U to subsets of $[0, 1]^3$, each triplet (T, I, F) typically satisfying $T + I + F \leq 3$ or a scaled version. In Definition 2.18, we have (T, I, F) with $T^q + I^q + F^q \leq 2$. If we set $q = 1$ and rescale the boundary appropriately (like $T + I + F \leq 2$ or a linear transformation to align with ≤ 3), we can embed an HNS. Formally:

$$\widetilde{Q}(x) = \tilde{A}(x) \quad \text{with the understanding that for each } (T, I, F) \in \tilde{A}(x), \ T + I + F \leq 2,$$

or we rescale so that $T^q + I^q + F^q \leq 2$ is equivalent to $T + I + F \leq 3$ after a linear or parametric transformation. Thus, ignoring the q-rung power or setting $q = 1$ collapses q-RHNS to an HNS. □

Definition 2.21 (q-Rung Orthopair n-SuperHyperNeutrosophic Set (q-RHNS$_n$)). Let U be a non-empty set, $q > 0$. Define recursively:

$$\mathcal{P}_1(U) = \mathcal{P}(U), \quad \mathcal{P}_k(U) = \mathcal{P}(\mathcal{P}_{k-1}(U)) \quad (k \geq 2).$$

Similarly, consider $\mathcal{P}_n([0, 1]^3)$ for the n-nested subsets of the unit cube $[0, 1]^3$. A *q-Rung Orthopair n-SuperHyperNeutrosophic Set (q-RHNS$_n$)* is a mapping

$$\widetilde{Q}_n : \mathcal{P}_n(U) \to \mathcal{P}_n([0, 1]^3),$$

such that for each $A \in \mathcal{P}_n(U)$, $\widetilde{Q}_n(A) \subseteq [0, 1]^3$ is a *set* of triplets (T, I, F) satisfying

$$T^q + I^q + F^q \leq 2.$$

Hence, each n-th level subset A is assigned a set of q-rung orthopair membership triplets in $[0, 1]^3$.

Theorem 2.22. *Every q-Rung Orthopair HyperNeutrosophic Set is a special case of a q-Rung Orthopair n-SuperHyperNeutrosophic Set for n = 1.*

Proof. A *q-Rung Orthopair HyperNeutrosophic Set (q-RHNS)* $\widetilde{Q}$ is a mapping $U \to \mathcal{P}([0,1]^3)$, each triplet satisfying (T, I, F) with $T^q + I^q + F^q \leq 2$. In Definition 2.21, for $n = 1$ we have:

$$\widetilde{Q}_1 : \mathcal{P}_1(U) = \mathcal{P}(U) \to \mathcal{P}_1([0,1]^3) = \mathcal{P}([0,1]^3).$$

We define:

$$\widetilde{Q}_1(\{x\}) := \widetilde{Q}(x), \quad \widetilde{Q}_1(A) = \varnothing \quad (\text{for } A \neq \{x\}).$$

Hence, for singletons $\{x\} \subset U$, we recover exactly the membership sets from $\widetilde{Q}(x)$. The q-rung condition remains. Thus, $\widetilde{Q}$ is embedded in $\widetilde{Q}_1$ as a special case. $\qquad\square$

Theorem 2.23. *Every n-SuperHyperNeutrosophic Set can be viewed as a special case of a q-Rung Orthopair n-SuperHyperNeutrosophic Set by letting q = 1 or ignoring the q-rung power.*

Proof. An *n-SuperHyperNeutrosophic Set (SHNS)* $\tilde{A}_n$ assigns each $A \in \mathcal{P}_n(U)$ a subset of $[0,1]^3$, each (T, I, F) satisfying $T + I + F \leq 3$ or a similar constraint. In Definition 2.21, a q-RHNS$_n$ uses the condition $T^q + I^q + F^q \leq 2$. If we set $q = 1$ and adjust the boundary from 2 to 3 by a simple scaling (or interpret sum ≤ 2 as a scaled version of ≤ 3), we recover the classical *n*-SHNS. Formally, define

$$\widetilde{Q}_n(A) = \tilde{A}_n(A) \quad \text{with the sum constraint replaced or scaled so } (T, I, F) \text{ meet } T^q + I^q + F^q \leq 2 \text{ for } q = 1.$$

Hence, ignoring or setting $q = 1$ collapses the q-rung approach to the usual sum-based approach. Therefore, each *n*-SHNS can be embedded in a q-RHNS$_n$ by suitably setting $q = 1$ and matching bounds. $\qquad\square$

2.4 Neutrosophic Overset, Underset, and Offset

Neutrosophic Overset, Underset, and Offset extend traditional neutrosophic sets. Overset includes external elements, Underset excludes specific elements, and Offset captures deviations, enhancing uncertainty and flexibility modeling [23, 65, 68, 78, 89, 90, 92, 95].

Definition 2.24 (Neutrosophic Overset). [81–83] Let U be a universe of discourse, and let $T(x), I(x), F(x)$ represent the truth, indeterminacy, and falsity membership functions, respectively. For a neutrosophic overset A, these functions satisfy:

$$T(x), I(x), F(x) : U \to [0, \Omega], \quad \Omega > 1.$$

A Neutrosophic Overset is given by:

$$A = \{(x, T(x), I(x), F(x)) \mid x \in U, \exists x \in U \text{ such that } \max(T(x), I(x), F(x)) > 1\}.$$

Definition 2.25 (Neutrosophic Underset). [81–83] Let U be a universe of discourse, and let $T(x), I(x), F(x)$ represent the truth, indeterminacy, and falsity membership functions, respectively. For a neutrosophic underset A, these functions satisfy:

$$T(x), I(x), F(x) : U \to [\Psi, 1], \quad \Psi < 0.$$

A Neutrosophic Underset is given by:

$$A = \{(x, T(x), I(x), F(x)) \mid x \in U, \exists x \in U \text{ such that } \min(T(x), I(x), F(x)) < 0\}.$$

Definition 2.26 (Neutrosophic Offset). [81–83] Let U be a universe of discourse, and let $T(x), I(x), F(x)$ represent the truth, indeterminacy, and falsity membership functions, respectively. For a neutrosophic offset A, these functions satisfy:

$$T(x), I(x), F(x) : U \to [\Psi, \Omega], \quad \Psi < 0, \quad \Omega > 1.$$

A Neutrosophic Offset is given by:

$$A = \{(x, T(x), I(x), F(x)) \mid x \in U, \exists x \in U \text{ such that } \min(T(x), I(x), F(x)) < 0 \text{ and } \max(T(x), I(x), F(x)) > 1\}.$$

Definition 2.27 (HyperNeutrosophic Overset/Underset/Offset)**.** Let U be a universe of discourse, and let $\Psi < 0 < 1 < \Omega$. Define intervals:

(Overset domain): $[0, \Omega]^3$, (Underset domain): $[\Psi, 1]^3$, (Offset domain): $[\Psi, \Omega]^3$.

A *HyperNeutrosophic Overset* (HNO) $\widetilde{A}$, *HyperNeutrosophic Underset* (HNU) $\widetilde{B}$, or *HyperNeutrosophic Offset* (HNOf) $\widetilde{C}$ is a mapping:

$$\widetilde{A} : U \to \mathcal{P}([0, \Omega]^3), \quad \widetilde{B} : U \to \mathcal{P}([\Psi, 1]^3), \quad \widetilde{C} : U \to \mathcal{P}([\Psi, \Omega]^3),$$

respectively, such that:

- *(Overset case):* There exists at least one $x \in U$ for which some $(T, I, F) \in \widetilde{A}(x)$ satisfies $\max\{T, I, F\} > 1$.

- *(Underset case):* There exists at least one $x \in U$ for which some $(T, I, F) \in \widetilde{B}(x)$ satisfies $\min\{T, I, F\} < 0$.

- *(Offset case):* There exists at least one $x \in U$ for which some $(T, I, F) \in \widetilde{C}(x)$ satisfies $\min\{T, I, F\} < 0$ and $\max\{T, I, F\} > 1$.

Hence, each element x is assigned a *set* of membership triples, possibly extending below 0 or above 1, depending on overset, underset, or offset definitions.

Theorem 2.28. *Every Neutrosophic Overset is a special case of a HyperNeutrosophic Overset.*

Proof. A *Neutrosophic Overset A* on U associates each $x \in U$ with one triple $(T(x), I(x), F(x))$ where $\max(T(x), I(x), F(x)) > 1$ for at least one x. In Definition 2.27, a *HyperNeutrosophic Overset $\widetilde{A}$* maps $x \in U$ to a *set* of $(T, I, F) \in [0, \Omega]^3$. We embed A by letting

$$\widetilde{A}(x) = \left\{\big(T(x), I(x), F(x)\big)\right\}$$

(a singleton). Thus, each element is assigned exactly one triple. The overset condition $\max\{T(x), I(x), F(x)\} > 1$ for some x remains, so A is recovered exactly as a degenerate (single membership) hyperneutrosophic overset. $\square$

Theorem 2.29. *Every HyperNeutrosophic Set is a special case of a HyperNeutrosophic Overset (resp. Underset, Offset) by restricting Ω to 1 (resp. Ψ to 0, $\Psi = 0, \Omega = 1$).*

Proof. A *HyperNeutrosophic Set $\widetilde{A}$* uses $[0, 1]^3$ for memberships. In the overset domain we have $[0, \Omega]^3$, with $\Omega > 1$. If we take $\Omega = 1$, that domain reverts to $[0, 1]^3$, so $\widetilde{A}$ is embedded trivially. The same logic applies to underset (set $\Psi = 0$) or offset (set $\Psi = 0, \Omega = 1$). Hence, ignoring the extended domain merges the set back into $[0, 1]^3$. $\square$

Theorem 2.30. *Every Neutrosophic Underset/Offset is a special case of a HyperNeutrosophic Underset/Offset, respectively.*

Proof. Parallel to Theorem 2.28, but for underset/offset. For an underset, we let $\widetilde{B}(x) = \{(T(x), I(x), F(x))\}$, a singleton in $[\Psi, 1]^3$, with $\min\{T, I, F\} < 0$ for at least one x. The offset proof is similar: $\widetilde{C}(x) = \{(T, I, F)\}$ with $\min < 0$ and $\max > 1$ for at least one x. Hence, singletons in the hyper domain replicate the single-valued case. $\square$

Definition 2.31 (n-SuperHyperNeutrosophic Overset/Underset/Offset)**.** Let U be a universe, and let $\Psi < 0 < 1 < \Omega$. For each $n \geq 1$, define $\mathcal{P}_n(U)$ as the n-th nested power set of U, and consider

$$\mathcal{P}_n([0, \Omega]^3), \quad \mathcal{P}_n([\Psi, 1]^3), \quad \mathcal{P}_n([\Psi, \Omega]^3)$$

for the overset, underset, and offset domains, respectively. Then:

- An *n-SuperHyperNeutrosophic Overset* $\widetilde{A}_n$ is a mapping

$$\widetilde{A}_n : \mathcal{P}_n(U) \to \mathcal{P}_n([0, \Omega]^3),$$

  with at least one $A \in \mathcal{P}_n(U)$ and some triple $(T, I, F) \in \widetilde{A}_n(A)$ such that $\max(T, I, F) > 1$.

- An *n-SuperHyperNeutrosophic Underset* $\widetilde{B}_n$ uses $\mathcal{P}_n([\Psi, 1]^3)$ with at least one $A \in \mathcal{P}_n(U)$ and some (T, I, F) where $\min(T, I, F) < 0$.

- An *n-SuperHyperNeutrosophic Offset* $\widetilde{C}_n$ uses $\mathcal{P}_n([\Psi, \Omega]^3)$ with at least one $A \in \mathcal{P}_n(U)$ and (T, I, F) where $\min(T, I, F) < 0$ and $\max(T, I, F) > 1$.

Thus, each n-th level subset is assigned a *set* of membership triples in the extended domain, capturing overset, underset, or offset behavior in an n-superhyper environment.

Theorem 2.32. *Every HyperNeutrosophic Overset (Underset, Offset) is a special case of an n-SuperHyperNeutrosophic Overset (Underset, Offset) for $n = 1$.*

Proof. Take the overset case for illustration (similar for underset/offset). Let $\widetilde{A}$ be a HyperNeutrosophic Overset mapping $U \to \mathcal{P}([0, \Omega]^3)$. In the n-super version, for $n = 1$ we have:

$$\widetilde{A}_1 : \mathcal{P}_1(U) = \mathcal{P}(U) \to \mathcal{P}_1([0, \Omega]^3) = \mathcal{P}([0, \Omega]^3).$$

Define $\widetilde{A}_1(\{x\}) := \widetilde{A}(x)$ and $\widetilde{A}_1(A) = \varnothing$ for $A \neq \{x\}$. Then singletons in $\mathcal{P}(U)$ recover exactly the membership sets $\widetilde{A}(x)$. The overset condition remains ($\max(T, I, F) > 1$ for some triple). Similarly for underset/offset. Hence, each HyperNeutrosophic overset/underset/offset is embedded in the n-super version with $n = 1$. □

Theorem 2.33. *Every Neutrosophic Overset (Underset, Offset) is a special case of an n-SuperHyperNeutrosophic Overset (Underset, Offset) by letting $n = 1$ and singletons.*

Proof. Parallels the logic in the previous theorems: we define $\widetilde{A}_n(\{x\}) = \{(T(x), I(x), F(x))\}$, a singleton, ensuring the overset/underset/offset condition is satisfied for at least one triple. This replicates the single-valued scenario in the n-superhyper context. □

Theorem 2.34. *Every n-SuperHyperNeutrosophic Set is a special case of an n-SuperHyperNeutrosophic Overset (Underset, Offset) by restricting Ω to 1 (resp. $\Psi = 0$, $\Psi = 0, \Omega = 1$).*

Proof. Same scaling or restriction arguments: if $\Omega = 1$ we lose the overset possibility above 1, if $\Psi = 0$ we lose negativity, etc. This recovers a normal n-SuperHyperNeutrosophic membership in $[0, 1]^3$. □

Funding

Acknowledgments

Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

References

[1] Muhammad Akram, A Nagoor Gani, and A Borumand Saeid. Vague hypergraphs. *Journal of Intelligent & Fuzzy Systems*, 26(2):647–653, 2014.

[2] Muhammad Irfan Ali. Another view on q-rung orthopair fuzzy sets. *International Journal of Intelligent Systems*, 33:2139 – 2153, 2018.

[3] Mumtaz Ali, Irfan Deli, and Florentin Smarandache. The theory of neutrosophic cubic sets and their applications in pattern recognition. *Journal of intelligent & fuzzy systems*, 30(4):1957–1963, 2016.

[4] Rasha Almajed, Watson Thompson, et al. Improving the perfoamnce of fog-assisted internet of things networks using bipolar trapezoidal neutrosophic sets. *International Journal of Wireless & Ad Hoc Communication*, 6(1), 2023.

[5] Krassimir Atanassov. Intuitionistic fuzzy sets. *International journal bioautomation*, 20:1, 2016.

[6] Krassimir Atanassov and George Gargov. Elements of intuitionistic fuzzy logic. part i. *Fuzzy sets and systems*, 95(1):39–52, 1998.

[7] Krassimir T Atanassov. *On intuitionistic fuzzy sets theory*, volume 283. Springer, 2012.

[8] Krassimir T Atanassov. Circular intuitionistic fuzzy sets. *Journal of Intelligent & Fuzzy Systems*, 39(5):5981–5986, 2020.

[9] Krassimir T Atanassov and Krassimir T Atanassov. *Intuitionistic fuzzy sets*. Springer, 1999.

[10] Krassimir T Atanassov and G Gargov. *Intuitionistic fuzzy logics*. Springer, 2017.

[11] Vivek Badhe, RS Thakur, and GS Thakur. Vague set theory for profit pattern and decision making in uncertain data. *International journal of advanced computer science and applications*, 6(6):58–64, 2015.

[12] Durga Banerjee, Bibhas C Giri, Surapati Pramanik, and Florentin Smarandache. Gra for multi attribute decision making in neutrosophic cubic set environment. *Neutrosophic Sets and Systems*, 15:60–69, 2017.

[13] Said Broumi, Mohamed Talea, Assia Bakali, and Florentin Smarandache. Shortest path problem under trapezoidal neutrosophic information. *Infinite Study*, 2017.

[14] Wen-Hua Cui and Jun Ye. Logarithmic similarity measure of dynamic neutrosophic cubic sets and its application in medical diagnosis. *Computers in Industry*, 111:198–206, 2019.

[15] Muhmamet Deveci, Dragan Pamucar, Ilgin Gokasar, Mario Köppen, and Brij Bhooshan Gupta. Personal mobility in metaverse with autonomous vehicles using q-rung orthopair fuzzy sets based opa-rafsi model. *IEEE Transactions on Intelligent Transportation Systems*, 24:15642–15651, 2023.

[16] Ronald C. Freiwald. An introduction to set theory and topology. 2014.

[17] Takaaki Fujita. A concise review on various concepts of superhyperstructures.

[18] Takaaki Fujita. Natural n-superhyper plithogenic language.

[19] Takaaki Fujita. Exploring concepts of hyperfuzzy, hyperneutrosophic, and hyperplithogenic sets. 2024. DOI: 10.13140/RG.2.2.12216.87045.

[20] Takaaki Fujita. Note for hypersoft filter and fuzzy hypersoft filter. *Multicriteria Algorithms With Applications*, 5:32–51, 2024.

[21] Takaaki Fujita. Note for neutrosophic incidence and threshold graph. *SciNexuses*, 1:97–125, 2024.

[22] Takaaki Fujita. Reconsideration of neutrosophic social science and neutrosophic phenomenology with non-classical logic. Technical report, Center for Open Science, 2024.

[23] Takaaki Fujita. A review of fuzzy and neutrosophic offsets: Connections to some set concepts and normalization function. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 74, 2024.

[24] Takaaki Fujita. Short communication of big plithogenic science and deep plithogenic science. 2024. preprint (researchgate).

[25] Takaaki Fujita. Short note of bunch graph in fuzzy, neutrosophic, and plithogenic graphs. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 294, 2024.

[26] Takaaki Fujita. Superhypergraph neural networks and plithogenic graph neural networks: Theoretical foundations. *arXiv preprint arXiv:2412.01176*, 2024.

[27] Takaaki Fujita. Survey of intersection graphs, fuzzy graphs and neutrosophic graphs. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 114, 2024.

[28] Takaaki Fujita. Survey of planar and outerplanar graphs in fuzzy and neutrosophic graphs. *ResearchGate*, July 2024.

[29] Takaaki Fujita. Uncertain labeling graphs and uncertain graph classes (with survey for various uncertain sets). July 2024.

[30] Takaaki Fujita. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*. Biblio Publishing, 2025.

[31] Takaaki Fujita. A comprehensive discussion on fuzzy hypersoft expert, superhypersoft, and indetermsoft graphs. *Neutrosophic Sets and Systems*, 77:241–263, 2025.

[32] Takaaki Fujita. Exploration of graph classes and concepts for superhypergraphs and n-th power mathematical structures. 2025.

[33] Takaaki Fujita. Exploring concepts of hyperfuzzy, hyperneutrosophic, and hyperplithogenic sets ii. *ResearchGate*, 2025.

[34] Takaaki Fujita. Hyperfuzzy hyperrough set, hyperneutrosophic hyperrough set, and hypersoft hyperrough set. Preprint, 2025.

[35] Takaaki Fujita. Short note of even-hole-graph for uncertain graph. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 351, 2025.

[36] Takaaki Fujita. Short note of extended hyperplithogenic sets, 2025. Preprint.

[37] Takaaki Fujita. A short note on the basic graph construction algorithm for plithogenic graphs. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 274, 2025.

[38] Takaaki Fujita. Some type of hyperneutrosophic set: Bipolar, pythagorean, double-valued, interval-valued set, 2025. Preprint.

[39] Takaaki Fujita. Some types of hyperneutrosophic set (2): Complex, single-valued triangular, fermatean, and linguistic sets. *Preprint*, 2025.

[40] Takaaki Fujita. Some types of hyperneutrosophic set (3): Dynamic, quadripartitioned, pentapartitioned, heptapartitioned, m-polar. 2025.

[41] Takaaki Fujita and Florentin Smarandache. A reconsideration of advanced concepts in neutrosophic graphs: Smart, zero divisor, layered, weak, semi, and chemical graphs. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 308.

[42] Takaaki Fujita and Florentin Smarandache. Antipodal turiyam neutrosophic graphs. *Neutrosophic Optimization and Intelligent Systems*, 5:1–13, 2024.

[43] Takaaki Fujita and Florentin Smarandache. Introduction to upside-down logic: Its deep relation to neutrosophic logic and applications. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond (Third Volume)*, 2024.

[44] Takaaki Fujita and Florentin Smarandache. Mixed graph in fuzzy, neutrosophic, and plithogenic graphs. *Neutrosophic Sets and Systems*, 74:457–479, 2024.

[45] Takaaki Fujita and Florentin Smarandache. A short note for hypersoft rough graphs. *HyperSoft Set Methods in Engineering*, 3:1–25, 2024.

[46] Takaaki Fujita and Florentin Smarandache. Study for general plithogenic soft expert graphs. *Plithogenic Logic and Computation*, 2:107–121, 2024.

[47] Takaaki Fujita and Florentin Smarandache. Uncertain automata and uncertain graph grammar. *Neutrosophic Sets and Systems*, 74:128–191, 2024.

[48] Takaaki Fujita and Florentin Smarandache. Local-neutrosophic logic and local-neutrosophic sets: Incorporating locality with applications. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 51, 2025.

[49] W-L Gau and Daniel J Buehrer. Vague sets. *IEEE transactions on systems, man, and cybernetics*, 23(2):610–614, 1993.

[50] Daniela Gifu. Soft sets extensions: Innovating healthcare claims analysis. *Applied Sciences*, 14(19):8799, 2024.

[51] Muhammad Gulistan, Ahmed Elmoasry, and Naveed Yaqoob. N-version of the neutrosophic cubic set: application in the negative influences of internet. *The Journal of Supercomputing*, 77(10):11410–11431, 2021.

[52] Karel Hrbacek and Thomas Jech. Introduction to set theory, revised and expanded. 2017.

[53] Hariwan Z Ibrahim. Multi-attribute group decision-making based on bipolar n, m-rung orthopair fuzzy sets. *Granular Computing*, 8(6):1819–1836, 2023.

[54] Hariwan Z Ibrahim. Multi-criteria decision-making based on similarity measures on interval-valued bipolar n, m-rung orthopair fuzzy sets. *Granular Computing*, 9(1):5, 2024.

[55] Thomas Jech. *Set theory: The third millennium edition, revised and expanded*. Springer, 2003.

[56] Young Bae Jun, Florentin Smarandache, and Chang Su Kim. Neutrosophic cubic sets. *New mathematics and natural computation*, 13(01):41–54, 2017.

[57] Hüseyin Kamacı, Harish Garg, and Subramanian Petchimuthu. Bipolar trapezoidal neutrosophic sets and their dombi operators with applications in multicriteria decision making. *Soft Computing*, 25(13):8417–8440, 2021.

[58] Muhammad Jabir Khan, Poom Kumam, and Meshal Shutaywi. Knowledge measure for the q-rung orthopair fuzzy sets. *International Journal of Intelligent Systems*, 36(2):628–655, 2021.

[59] Kiran Khatter. Interval valued trapezoidal neutrosophic set: multi-attribute decision making for prioritization of non-functional requirements. *Journal of Ambient Intelligence and Humanized Computing*, 12(1):1039–1055, 2021.

[60] Kazimierz Kuratowski. Introduction to set theory and topology. 1964.

[61] BongJu Lee and Yong Sik Yun. The generalized trapezoidal fuzzy sets. *Journal of the chungcheong mathematical society*, 24(2):253–266, 2011.

[62] Donghai Liu, Xiao hong Chen, and Dan Peng. Some cosine similarity measures and distance measures between q-rung orthopair fuzzy sets. *International Journal of Intelligent Systems*, 34:1572 – 1587, 2019.

[63] An Lu and Wilfred Ng. Vague sets or intuitionistic fuzzy sets for handling vague data: which one is better? In *International conference on conceptual modeling*, pages 401–416. Springer, 2005.

[64] Pradip Kumar Maji, Ranjit Biswas, and A Ranjan Roy. Soft set theory. *Computers & mathematics with applications*, 45(4-5):555–562, 2003.

[65] Nivetha Martin, Priya Priya.R, and Florentin Smarandache. Decision making on teachers' adaptation to cybergogy in saturated interval- valued refined neutrosophic overset /underset /offset environment. *International Journal of Neutrosophic Science*, 2020.

[66] Jerry M Mendel. On computing the similarity of trapezoidal fuzzy sets using an automated area method. *Information Sciences*, 589:716–737, 2022.

[67] Dmitriy Molodtsov. Soft set theory-first results. *Computers & mathematics with applications*, 37(4-5):19–31, 1999.

[68] p. geetha and K. Anitha. Single valued neutrosophic soft over / under / offsets. 2016.

[69] Xindong Peng and Lin Liu. Information measures for q-rung orthopair fuzzy sets. *International Journal of Intelligent Systems*, 34:1795 – 1834, 2019.

[70] Surapati Pramanik, Shyamal Dalapati, Shariful Alam, and Tapan Kumar Roy. Nc-todim-based magdm under a neutrosophic cubic set environment. *Information*, 8(4):149, 2017.

[71] Pratibha Rani and Arunodaya Raj Mishra. Multi-criteria weighted aggregated sum product assessment framework for fuel technology selection using q-rung orthopair fuzzy sets. *Sustainable Production and Consumption*, 24:90–104, 2020.

[72] Vanita Rani and Satish Kumar. An innovative distance measure for quantifying the dissimilarity between q-rung orthopair fuzzy sets. *Decision Analytics Journal*, 11:100440, 2024.

[73] Judith Roitman. *Introduction to modern set theory*, volume 8. John Wiley & Sons, 1990.

[74] Sovan Samanta, Madhumangal Pal, Hossein Rashmanlou, and Rajab Ali Borzooei. Vague graphs and strengths. *Journal of Intelligent & Fuzzy Systems*, 30(6):3675–3680, 2016.

[75] S Santhoshkumar, J Aldring, and D Ajay. Analyzing aggregation operators on complex q-rung orthopair neutrosophic sets with their application. In *International Conference on Intelligent and Fuzzy Systems*, pages 744–751. Springer, 2024.

[76] F. Smarandache. Introduction to superhyperalgebra and neutrosophic superhyperalgebra. *Journal of Algebraic Hyperstructures and Logical Algebras*, 2022.

[77] Florentin Smarandache. Ambiguous set (as) is a particular case of the quadripartitioned neutrosophic set (qns). *nidus idearum*, page 16.

[78] Florentin Smarandache. Neutrosophic overset, neutrosophic underset, and neutrosophic offset. similarly for neutrosophic over-/under-/offlogic, probability, and statisticsneutrosophic, pons editions brussels, 170 pages book, 2016.

[79] Florentin Smarandache. Neutrosophy: neutrosophic probability, set, and logic: analytic synthesis & synthetic analysis. 1998.

[80] Florentin Smarandache. A unifying field in logics: Neutrosophic logic. In *Philosophy*, pages 1–141. American Research Press, 1999.

[81] Florentin Smarandache. Degrees of membership¿ 1 and¡ 0 of the elements with respect to a neutrosophic offset. *Neutrosophic Sets and Systems*, 12:3–8, 2016.

[82] Florentin Smarandache. Neutrosophic overset. *Neutrosophic Underset, and Neutrosophic Offset: Similarly for Neutrosophic Over-/Under-/Off-Logic, Probability, and Statistics, Pons Editions Brussels*, 2016.

[83] Florentin Smarandache. Neutrosophic overset applied in physics. In *69th Annual Gaseous Electronics Conference, Bochum, Germany [through American Physical Society (APS)]*, 2016.

[84] Florentin Smarandache. Hyperuncertain, superuncertain, and superhyperuncertain sets/logics/probabilities/statistics. *Critical Review*, XIV, 2017.

[85] Florentin Smarandache. *Plithogeny, plithogenic set, logic, probability, and statistics*. Infinite Study, 2017.

[86] Florentin Smarandache. Extension of soft set to hypersoft set, and then to plithogenic hypersoft set. *Neutrosophic sets and systems*, 22(1):168–170, 2018.

[87] Florentin Smarandache. *Plithogenic set, an extension of crisp, fuzzy, intuitionistic fuzzy, and neutrosophic sets-revisited*. Infinite study, 2018.

[88] Florentin Smarandache. Plithogeny, plithogenic set, logic, probability, and statistics. *arXiv preprint arXiv:1808.03948*, 2018.

[89] Florentin Smarandache. *Practical Applications of the Independent Neutrosophic Components and of the Neutrosophic Offset Components*. Infinite Study, 2021.

[90] Florentin Smarandache. Interval-valued neutrosophic oversets, neutrosophic undersets, and neutrosophic offsets. *Collected Papers. Volume IX: On Neutrosophic Theory and Its Applications in Algebra*, page 117, 2022.

[91] Florentin Smarandache. Foundation of superhyperstructure & neutrosophic superhyperstructure. *Neutrosophic Sets and Systems*, 63(1):21, 2024.

[92] Florentin Smarandache. Operators for uncertain over/under/off-sets/-logics/probabilities/-statistics. *Neutrosophic Sets and Systems*, 79:493–500, 2025.

[93] Florentin Smarandache and Said Broumi. *Neutrosophic graph theory and algorithms*. IGI Global, 2019.

[94] Florentin Smarandache and NM Gallup. Generalization of the intuitionistic fuzzy set to the neutrosophic set. In *International Conference on Granular Computing*, pages 8–42. Citeseer, 2006.

[95] Florentin Smarandache, Miguel A. Quiroz-Martínez, Jesús Estupiñán, Noel Batista Hernández, and Maikel Yelandi Leyva Vazquez. Application of neutrosophic offsets for digital image processing. 2020.

[96] Metawee Songsaeng, Aiyared Iampan, et al. *Neutrosophic cubic set theory applied to UP-algebras*. PhD thesis, University of Phayao, 2024.

[97] Samajh Singh Thakur and Saeid Jafari. (m, a, n)-fuzzy neutrosophic sets and their topological structure. *Neutrosophic Sets and Systems*, 73:592–605, 2024.

[98] Michael Gr Voskoglou, Florentin Smarandache, and Mona Mohamed. *q-Rung Neutrosophic Sets and Topological Spaces*. Infinite Study, 2024.

[99] Ping Wang, Jie Wang, Guiwu Wei, and Cun Wei. Similarity measures of q-rung orthopair fuzzy sets based on cosine function and their applications. *Mathematics*, 2019.

[100] Zhi Xiao, Sisi Xia, Ke Gong, and Dan Li. The trapezoidal fuzzy soft set and its application in mcdm. *Applied Mathematical Modelling*, 36:5844–5855, 2012.

[101] Jun Ye. Trapezoidal neutrosophic set and its application to multiple attribute decision-making. *Neural computing and Applications*, 26:1157–1166, 2015.

[102] P Yiarayong. Some weighted aggregation operators of quadripartitioned single-valued trapezoidal neutrosophic sets and their multi-criteria group decision-making method for developing green supplier selection criteria. *OPSEARCH*, pages 1–55, 2024.

[103] YS Yun. Parametric operations for two 2-dimensional trapezoidal fuzzy sets. *Journal of Algebra & Applied Mathematics*, 18(1), 2020.

[104] YS Yun. Parametric operations between 3-dimensional triangular fuzzy number and trapezoidal fuzzy set. *Journal of Algebra & Applied Mathematics*, 21(2), 2023.

[105] Lotfi A Zadeh. Fuzzy sets. *Information and control*, 8(3):338–353, 1965.

[106] Lotfi A Zadeh. A fuzzy-set-theoretic interpretation of linguistic hedges. 1972.

[107] Lotfi A Zadeh. Fuzzy sets and their application to pattern classification and clustering analysis. In *Classification and clustering*, pages 251–299. Elsevier, 1977.

[108] Lotfi A Zadeh. Fuzzy sets versus probability. *Proceedings of the IEEE*, 68(3):421–421, 1980.

[109] Lotfi A Zadeh. Fuzzy logic, neural networks, and soft computing. In *Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers by Lotfi A Zadeh*, pages 775–782. World Scientific, 1996.

[110] Jianming Zhan, Madad Khan, Muhammad Gulistan, and Ahmed Ali. Applications of neutrosophic cubic sets in multi-criteria decision-making. *International Journal for Uncertainty Quantification*, 7(5), 2017.